ON A TWO VARIABLE CLASS OF BERNSTEIN-SZEGŐ MEASURES

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Abstract. The one variable Bernstein-Szegő theory for orthogonal polynomials on the real line is extended to a class of two variable measures. The polynomials orthonormal in the total degree ordering and the lexicographical ordering are constructed and their recurrence coefficients discussed.

1. Introduction

Let $\mu(x,y)$ be a Borel measure supported on $\mathbb{R}^2$, such that

$$\int_{\mathbb{R}^2} f(x,y)d\mu(x,y) < \infty$$

for every polynomial $f(x,y)$. We would like to construct polynomials orthogonal with respect to this measure. Unlike one variable there are many natural orderings of the monomials $x^n y^m$ and each of these orderings gives a different set of orthogonal polynomials. Starting with [7], the preferred ordering is the total degree ordering and for polynomials with the same total degree the ordering is lexicographical, that is

$$(k,l) <_{\mathrm{td}} (k_1,l_1)$$

if

$$k+l < k_1+l_1 \text{ or } (k+l = k_1+l_1 \ (k,l) <_{\mathrm{lex}} (k_1,l_1)),$$

where lex means the lexicographical ordering (see below). In other words, we apply the Gram-Schmidt process to the polynomials ordered as follows $\{1, y, x, y^2, xy, x^2, \dots\}$. Let p_n^r denote the polynomial of total degree n such that

$$p_n^r(x,y) = k_{n,r}^{r,n-r} x^r y^{n-r} + \sum_{(i,j)<_{\mathrm{td}}(r,n-r)} k_{n,r}^{i,j} x^i y^j, \tag{1.1}$$

with $k_{n,r}^{r,n-r} > 0$ satisfying

$$\begin{aligned} &\int_{\mathbb{R}^2} p_n^r x^i y^j d\mu(x,y) = 0, \quad 0 \le i+j < n \text{ or } i+j = n \text{ and } 0 \le i < r, \\ &\int_{\mathbb{R}^2} p_n^r p_n^r d\mu(x,y) = 1. \end{aligned} \tag{1.2}$$

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Let $\mathbf{P}_n(x,y)$ be the $(n+1)$ dimensional vector with components the orthonormal polynomials of total degree n. The multiplications by x and y are given by three term recurrence relations

$$x\mathbf{P}_n = A_{x,n}\mathbf{P}_{n+1} + B_{x,n}\mathbf{P}_n + A^t_{x,n-1}\mathbf{P}_{n-1} \tag{1.3}$$

$$y\mathbf{P}_n = A_{y,n}\mathbf{P}_{n+1} + B_{y,n}\mathbf{P}_n + A^t_{y,n-1}\mathbf{P}_{n-1}. \tag{1.4}$$

where $A_{x,n}$, $A_{y,n}$ are $(n+1)\times(n+2)$ matrices such that $\operatorname{rank}(A_{x,n}) = \operatorname{rank}(A_{y,n}) = n+1$, $B_{x,n}$, $B_{y,n}$ are symmetric $(n+1)\times(n+1)$ matrices and M^t denotes the transpose of the matrix M. Notice that the block Jacobi matrices corresponding to multiplications by x and y will commute which amounts to certain commutativity relations between the matrices defined above, see [4] for more details. Since monomials with the same total degree are ordered lexicographically we see that $A_{y,n}$ is "lower triangular" with positive entries in the (i,i), $i = 1\ldots,n+1$ positions and $A_{x,n}$ is lower Hessenberg with positive entries in $(i,i+1)$, $i = 1,\ldots,n+1$ entries.

Recently an alternative way to approach two dimensional orthogonal polynomials was proposed in [2] by relating them to the theory of matrix valued orthogonal polynomials. This can be accomplished by using the lexicographical ordering,

$$(k,l) <_{\text{lex}} (k_1,l_1) \Leftrightarrow k < k_1 \text{ or } (k = k_1 \text{ and } l < l_1),$$

or the reverse lexicographical ordering

$$(k,l) <_{\text{revlex}} (k_1,l_1) \Leftrightarrow (l,k) <_{\text{lex}} (l_1,k_1),$$

to arrange the monomials. This naturally connects the theory of bivariate orthogonal polynomials to doubly Hankel matrices, that is block Hankel matrices whose entries are Hankel matrices.

For every nonnegative integer m we apply the Gram-Schmidt process to the basis of monomials ordered in lexicographical order i.e.

$$\{1, y, \ldots, y^m, x, xy, \ldots, xy^m, x^2, \ldots\},$$

and define the orthonormal polynomials $p^l_{n,m}(x,y)$ where $0 \le l \le m$, by the equations,

(1.5)

$$\int_{\mathbb{R}^2} p^l_{n,m} x^i y^j \, d\mu(x,y) = 0, \quad 0 \le i < n \text{ and } 0 \le j \le m \;\text{ or } i = n \text{ and } 0 \le j < l,$$

$$\int_{\mathbb{R}^2} p^l_{n,m} p^l_{n,m} \, d\mu(x,y) = 1,$$

and

$$p^l_{n,m}(x,y) = k^{n,l}_{n,m,l} x^n y^l + \sum_{(i,j) <_{\text{lex}} (n,l)} k^{i,j}_{n,m,l} x^i y^j. \tag{1.6}$$

With the convention $k^{n,l}_{n,m,l} > 0$, the above equations uniquely specify $p^l_{n,m}$. Polynomials orthonormal with respect to $d\mu$ but using the reverse lexicographical ordering will be denoted by $\tilde{p}^l_{n,m}$. They are uniquely determined by the above relations with the roles of n and m interchanged. Set

$$\mathbb{P}_{n,m} = \begin{bmatrix} p^0_{n,m} \\ p^1_{n,m} \\ \vdots \\ p^m_{n,m} \end{bmatrix}. \tag{1.7}$$

The polynomials $\mathbb{P}_{n,m}$ may be obtained in an alternate manner as follows. We associate an $(m+1)\times(m+1)$ matrix valued measure $dM^{m+1}(x)$ by taking

$$dM^{m+1}(x) = \int_{\mathbb{R}} [1, y, \dots, y^m]^t d\mu(x,y)[1, y, \dots, y^m], \tag{1.8}$$

where the above integral is with respect to y. Let us denote by $\{P_n^m\}_{n=0}^{\infty}$ the sequence of $(m+1)\times(m+1)$ matrix valued polynomials satisfying

$$P_n^m(x) = K_{n,n}^m x^n + \text{lower order terms}, \tag{1.9}$$

$$\int_{\mathbb{R}} P_n^m(x) dM^{m+1}(P_k^m(x))^t = \delta_{k,n} I_{m+1} \tag{1.10}$$

with $K_{n,n}^m$ a lower triangular matrix with strictly positive diagonal entries. The above conditions uniquely specify these left matrix valued orthogonal polynomials and it follows that

$$\mathbb{P}_{n,m}(x,y) = P_n^m(x)[1, y, \dots, y^m]^t. \tag{1.11}$$

From the relation between $\mathbb{P}_{n,m}$ and the matrix orthogonal polynomials we see that the following recurrence formulas hold

$$x\mathbb{P}_{n,m} = A_{n+1,m}\mathbb{P}_{n+1,m} + B_{n,m}\mathbb{P}_{n,m} + A_{n,m}^t\mathbb{P}_{n-1,m}, \tag{1.12}$$

where $A_{n,m}$ and $B_{n,m}$ are matrices of size $(m+1)\times(m+1)$ given by,

$$A_{n,m} = \langle x\mathbb{P}_{n-1,m}, \mathbb{P}_{n,m}\rangle, \tag{1.13}$$

$$B_{n,m} = \langle x\mathbb{P}_{n,m}, \mathbb{P}_{n,m}\rangle. \tag{1.14}$$

Here $A_{n,m}$ is lower triangular with positive diagonal entries and $B_{n,m}$ is symmetric. The analogous formula for $\tilde{\mathbb{P}}_{n,m}(x,y)$ is

$$y\tilde{\mathbb{P}}_{n,m} = \tilde{A}_{n,m+1}\tilde{\mathbb{P}}_{n,m+1} + \tilde{B}_{n,m}\tilde{\mathbb{P}}_{n,m} + \tilde{A}_{n,m}^t\tilde{\mathbb{P}}_{n,m-1}. \tag{1.15}$$

There has been much work done in studying bivariate orthogonal polynomials in the the total degree ordering and we refer to the books [1, 4, 8], and references therein. For Bernstein-Szegő polynomials related to root systems see the newer article [10].

In this article we will consider a special class of “Bernstein-Szegő” measures (see below for the definition) which are generalizations of the examples considered in [5] and construct orthogonal polynomials where the orderings used will be the total degree ordering or the lexicographical ordering. We will do this in order to investigate and compare the properties of the polynomials constructed. The paper is organized as follows: In section 2 we briefly review the one variable Bernstein-Szegő theory. We then consider two variable generalizations of this theory in section 3. Orthonormal polynomials in the total degree ordering as well as the lexicographical and reverse lexicographical orderings are considered. In section 4 we investigate the properties of the coefficients in the recurrence formulas satisfied by these polynomials and in section 5 we consider several examples including the ones discussed in [5].

2. One variable Bernstein-Szegő measures

Consider a measure supported on $[-1,1]$ absolutely continuous with respect to Lebesgue measure of the form $d\mu(x)=w(x)dx$ with $w(x)=(1-x^2)^{1/2}/\rho(x)$ where $\rho(x)$ is a polynomial of degree l positive on $[-1,1]$. Such measures are in the class of Bernstein-Szegő measures which have many nice properties. We call a polynomial $h(z)$ stable if $h(z)\neq 0,\ |z|\leq 1$. We have [9]

Theorem 2.1. *Let $w(x)=(1-x^2)^{1/2}/\rho(x)$ where $\rho(x)$ is a polynomial of exact degree N and is positive on $[-1,1]$. Let $\rho(x)=|h(e^{i\theta})|^2$ where $h(z)$ is a stable polynomial of exact degree N in z such that $h(0)>0$. Then*

$$p_n(x)=\left(\frac{2}{\pi}\right)^{1/2}(\sin\theta)^{-1}\Im(e^{i(n+1)\theta}\bar{h}(e^{i\theta})),\qquad x=\cos\theta,\ N<2(n+1),$$

is an orthonormal polynomial of degree n associated with w.

The above Theorem shows that the orthogonal polynomials associated with Bernstein-Szegő weights can for large enough n be written as a linear combinations of Chebyshev polynomials (for instance [6]) . A proof of the above Theorem in the case where the coefficients of h are real is given in Lemma 3.1. For real coefficients the above formula holds for $N=2(n+1)$ if the right hand side is multiplied by $(1-h_N/h_0)^{-1/2}$, where $h_0=h(0)$ and h_N is the coefficient of z^N in $h(z)$ We now show how to construct orthogonal polynomials of degree $\leq\lceil\frac{N-2}{2}\rceil-1$.

Proposition 2.2. *Assume that $h(z)=1+h_1z+\cdots+h_Nz^N$ is a stable polynomial of degree N with real coefficients. Let q_k be defined by*

$$q_k(x)=\sum_{i=0}^{N}h_iU_{k-i}(x),\qquad k\geq 0,\tag{2.1}$$

where $U_n(x)=-U_{-n-2}(x)$ for $n<0$. For $0\leq k\leq\lceil\frac{N-2}{2}\rceil-1$, there are constants h'_k such that

$$\hat{q}_k(x):=q_k(x)+h'_{k+1}q_{k+1}(x)+\cdots+h'_{N-k+2}q_{N-k-2}(x)$$

is a polynomial of degree k and orthogonal to every polynomial of degree less than k with respect to

$$d\mu(x)=\frac{\sqrt{1-x^2}dx}{|h(z)|^2},\qquad x=(z+1/z)/2,$$

where $h'_{N-k+2}=h_N$, $h'_{N-k+1}=h_{N-1}-h_Nh_1$, and the other h'_j can be deduced inductively from h_j.

Proof. Lemma 3.1 shows that q_k is orthogonal to all polynomials of degree at most $k-1$. Thus, it follows readily that for $0\leq k\leq\lceil\frac{N-2}{2}\rceil-1$, $\hat{q}_k$ is orthogonal to all polynomials of degree at most $k-1$. We now prove that we can choose constants h'_j such that $\hat{q}_k$ is of degree k. Using $U_n(x)=-U_{-n-2}(x)$ for $n<0$, we can write

$$q_k(x)=-\sum_{i=k+1}^{N-k-2}h_{k+i+2}U_i(x)+\sum_{i=0}^{k}(h_{k-i}-h_{k+i+2})U_i(x),\qquad k\leq\left\lceil\frac{N-2}{2}\right\rceil-1,\tag{2.2}$$

in which the first sum contains terms that have degree $> k$. In particular, $\deg q_k \leq N-k-2$ for $k \leq \lceil \frac{N-2}{2} \rceil - 1$. Furthermore, for $k+1 \leq j \leq N-1$ we can write, by (2.1) and the fact that $U_n(x) = -U_{-n-2}(x)$ for $n < 0$,

$$q_j(x) = \sum_{i=k+1}^{j} h_{j-i} U_i(x) + \sum_{i=0}^{k} h_{j-i} U_i(x) - \sum_{i=0}^{N-j-2} h_{j+i+2} U_i(x),$$

where again the first sum contains terms that have degree $> k$ in x. Since $h_0(y) = 1$, it follows readily that

$$q_k(x) + h_N q_{N-k-2}(x) = -\sum_{i=k+1}^{N-k-3} \left(h_{k+i+2} - h_N h_{N-k-2-i}\right) U_i(x) + \cdots,$$

where only terms whose degree $\geq k+1$ are given explicitly in the right hand side. Thus the right hand side of the above expression has degree $N-k-3$. Continuing, we add $(h_{N-1} - h_N h_1) q_{N-k-3}$ to eliminate U_{N-k-3}. Proceeding in this way, we keep adding terms until the right hand side contains only terms of degree $\leq k$. This proves that $\hat{q}_k$ is indeed a polynomial of degree k. □

Example 2.3. The first non-trivial case of Theorem 2.1 and Proposition 2.2 appears when $N = 5$ for which $\lceil \frac{N-2}{2} \rceil = 2$ and we are missing the orthogonal polynomial of first order. In this case, the formula (2.1) shows that

$$q_1(x) = -h_5 U_2(x) + (h_0 - h_4) U_1(x) + (h_1 - h_3) U_0(x)$$
$$q_2(x) = h_0 U_2(x) + (h_1 - h_5) U_1(x) + (h_2 - h_4) U_0(x).$$

Our $\hat{q}_1$ is given by

$$\hat{q}_1(x) = q_1(x) + h_5 q_2(x) = [(h_1 - h_3) + h_5(h_2 - h_4)] U_1(x) + [(h_0 - h_4) + h_5(h_1 - h_5)] U_0(x),$$

indeed a polynomial of degree 1.

Remark 2.4. The above proposition shows how a complete orthogonal basis can be derived for the Bernstein-Szegő weight function. A similar approach will be used in the construction of bivariate polynomials orthogonal in the lexicographical ordering. However, this procedure will not help us with the construction of bivariate polynomials orthogonal in total degree ordering. Indeed, when h_k are polynomials of y, the degree of $\hat{q}_k$ as a polynomial of two variables could have total degree greater than k.

We will also investigate the recurrence coefficients associated with Bernstein-Szegő weights. Let $\mu(x)$ be a positive measure supported on the real line with an infinite number of points of increase and $\{p_m\}$ be the set of polynomials of degree m orthonormal with respect to μ, each with positive leading coefficient. Then it is well known that they satisfy the recurrence relation

$$x p_m(x) = a_{m+1} p_{m+1}(x) + b_m p_m(x) + a_m p_{m-1}(x).$$

If the measure μ is of the form considered in Theorem 2.1 then the formula for p_m above shows that for $l < 2m$ they satisfy the same recurrence formula as the Chebyshev polynomials i.e $a_m = 1/2$ and $b_m = 0$. For extensions of this result see [3].

3. A class of Two Variable Bernstein-Szegő polynomials.

Here we study two variable polynomials orthogonal with respect to a Bernstein-Szegő weight of the form

$$\sigma(x,y) = \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{\rho(x,y)},$$

where ρ is a polynomial in x and y, positive for $-1 \le x \le 1$ and $-1 \le y \le 1$.

Lemma 3.1. *Let $N \in \mathbb{N}$ be fixed positive integer. For any $i = 0, \dots, N$ let $h_i(y)$ be polynomials in y with real coefficients of degree at most $\frac{N}{2} - |\frac{N}{2} - i|$, with $h_0(y) = 1$, such that*

$$h(z,y) = \sum_{i=0}^{N} h_i(y) z^i, \tag{3.1}$$

is a stable polynomial in z for all $-1 \le y \le 1$, i.e. $h(z,y) \neq 0$ for any $|z| \le 1$.

Define

$$q_k(x,y) = \sum_{i=0}^{N} h_i(y) U_{k-i}(x), \tag{3.2}$$

where $U_n(x)$ is the n-th Chebyshev polynomial of the second kind. Here, if $n < 0$ the Chebyshev polynomial is understood as $U_n(x) = -U_{-n-2}(x)$.

Then $q_k(x,y)$ is a polynomial in two variables, which is orthogonal to every polynomial in x of degree less than k with respect to

$$d\mu_y(x) = \frac{\sqrt{1-x^2}}{|h(z,y)|^2} dx, \qquad x = (z+1/z)/2.$$

Moreover for $k \ge \lceil \frac{N-2}{2} \rceil$, $q_k(x,y)$ is a polynomial of total degree k and

$$\int_{-1}^{1} q_k^2(x,y) d\mu_y(x) = \begin{cases} \frac{\pi}{2} & \text{if } k \ge \lceil \frac{N-1}{2} \rceil \\ \frac{\pi}{2}(1-h_N) & \text{if } 2k+2 = N \end{cases}.$$

Proof. We begin by showing that q_k is of total degree k when $k \ge \lceil \frac{N-2}{2} \rceil$. Note that $|h_N(y)| < 1$ follows from the stability of h. If $k \ge N$, then q_k is of degree k follows obviously from $\frac{N}{2} - |\frac{N}{2} - i| \le i$. Let now $k \le N-1$. Then using the fact that $U_{-j-2}(x) = -U_j(x)$ for $j < 0$, we have

$$q_k(x,y) = \sum_{i=0}^{k} h_i(y) U_{k-i}(x) - \sum_{i=k+2}^{N} h_i(y) U_{i-k-2}(x). \tag{3.3}$$

It is easy to see that $\frac{N}{2} - |\frac{N}{2} - i| + i - k - 2 \le N - k - 2 \le k$ as $k \ge \lceil \frac{N-2}{2} \rceil$. Hence, q_k is of degree k.

To establish the claimed orthogonality properties of q_k we follow Szegő. Since $U_n(x) = \frac{z^{n+1} - z^{-n-1}}{z - 1/z}$, the polynomials $q_k(x,y)$ can be written in the following form

$$\begin{aligned} q_k(x,y) &= \frac{1}{z-1/z} \sum_{i=0}^{N} h_i(y)(z^{k-i+1} - z^{-k+i-1}) \\ &= \frac{1}{z-1/z} \left[z^{k+1} h(1/z,y) - z^{-k-1} h(z,y) \right], \end{aligned}$$

and this identity holds for every k.

Take $j < k$ and integrate $q_k(x,y)U_j(x)$. Substituting $x = \cos\theta$ we get

$$\begin{aligned}
&\int_{-1}^{1} q_k(x,y)U_j(x)\frac{\sqrt{1-x^2}}{|h(z,y)|^2}dx \\
&\qquad = \int_0^\pi \frac{z^{k+1}h(1/z,y) - z^{-k-1}h(z,y)}{(z-1/z)^2 h(z,y)h(1/z,y)}(z^{j+1} - z^{-j-1})\sin^2\theta d\theta \\
&\qquad = \int_0^\pi \frac{z^{k+1}(z^{j+1} - z^{-j-1})}{(z-1/z)^2 h(z,y)}\sin^2\theta d\theta - \int_0^\pi \frac{z^{-k-1}(z^{j+1} - z^{-j-1})}{(z-1/z)^2 h(1/z,y)}\sin^2\theta d\theta.
\end{aligned}$$

If θ is replaced by $-\theta$ in the second integral, the two integrals can be combined into one integral over $[-\pi,\pi]$. This can be rewritten as a contour integral over $|z| = 1$, with $z = e^{i\theta}$ as follows

$$\int_{-\pi}^{\pi} \frac{z^{k+1}(z^{j+1} - z^{-j-1})}{h(z,y)}\frac{\sin^2\theta}{(z-1/z)^2}d\theta = \frac{i}{4}\int_{|z|=1}\frac{z^{k+j+1} - z^{k-j-1}}{h(z,y)}dz. \tag{3.4}$$

Since $h(z,y)$ is a stable polynomial the last integral vanishes for $k > j$ by the residue theorem.

Finally, let us see that for $k \geq \lceil \frac{N-2}{2} \rceil$ the norm of q_k does not depend upon y. First let $N < 2k+2$. Using (3.4) we obtain

$$\begin{aligned}
\int_{-1}^{1} q_k(x,y)q_k(x,y)\frac{\sqrt{1-x^2}}{|h(z,y)|^2}dx &= \int_{-1}^{1} q_k(x,y)h_0(y)U_k(x)\frac{\sqrt{1-x^2}}{|h(z,y)|^2}dx \\
&= \frac{ih_0(y)}{4}\int_{|z|=1}\frac{z^{2k+1} - z^{-1}}{h(z,y)}dz = \frac{-ih_0(y)}{4}\int_{|z|=1}\frac{z^{-1}}{h(z,y)}dz \\
&= \frac{\pi}{2}\frac{h_0(y)}{h(0,y)} = \frac{\pi}{2}.
\end{aligned}$$

For the case $N = 2k+2$, (3.3) shows that $q_k(x,y) = (h_0 - h_N)U_k(x) + \ldots$, so that a simple modification of the previous computation gives that the norm of q_k is $\frac{\pi}{2}(1 - h_N(y)/h_0(0,y)) = \frac{\pi}{2}(1 - h_N)$. □

Remark 3.2. The condition $\deg h_i(y) \leq \frac{N}{2} - |\frac{N}{2} - i|$ in Lemma 3.1 is sufficient for the class of Bernstein-Szegő measures considered in the present paper. However, the statement can be easily extended to more general weights. For instance, if $\deg h_i \leq i$ for every $i = 0,1,\ldots,N$, then $q_k(x,y)$ is a polynomial of total degree k in x and y for $k \geq N-1$.

Let Π_n denote the space of polynomials of total degree at most n and $\Pi_{n,m} = \operatorname{span}\{x^i y^j, 0 \leq i \leq n, 0 \leq j \leq m\}$. Using the above Lemma, we can construct polynomials of total degree n orthogonal to all polynomials in Π_{n-1}.

Theorem 3.3. *Consider the two variable measure*

$$d\mu(x,y) = \frac{4}{\pi^2}\frac{\sqrt{1-x^2}\sqrt{1-y^2}}{|h(z,y)|^2}\,dx\,dy, \tag{3.5}$$

where $h(z,y)$ is as in (3.1). *Then, for $\lceil \frac{N-2}{2} \rceil \leq k \leq n$, the polynomial*

$$p_n^k(x,y) = q_k(x,y)U_{n-k}(y)$$

is orthogonal to all polynomials of degree less than $(k, n-k)$.

Proof. From Lemma 3.1 it follows that $q_k(x,y)U_{n-k}(y)$, $\lceil\frac{N-2}{2}\rceil \leq k \leq n$ is of degree $(k,n-k)$ in Π_n. The result follows directly from previous lemma and the orthogonality of Chebyshev polynomials. To see this consider the product $U_i(x)U_j(y)$ if $i<k, i+j\leq n$ and $q_i(x,y)U_j(y)$ if $i\geq k, i+j<n$. Then, integrating first with respect to x and using the previous lemma yields,

$$\int_{-1}^{1}\int_{-1}^{1} p_n^k(x,y)U_i(x)U_j(y)d\mu(x,y)=0,$$

if $i<k$ and also

$$\int_{-1}^{1}\int_{-1}^{1} p_n^k(x,y)q_i(x,y)U_j(y)d\mu(x,y)=0,$$

for $i>k$. Finally, for $i=k$ we perform the integration with respect to x, and since the norm of $q_k(x,y)$ does not depend on y, the orthogonality of Chebyshev polynomials shows that for $k+j\leq n$,

$$\int_{-1}^{1}\int_{-1}^{1} p_n^k(x,y)q_k(x,y)U_j(y)d\mu(x,y)=\frac{2}{\pi}\int_{-1}^{1}U_{n-k}(y)U_j(y)\sqrt{1-y^2}\,dy=\delta_{n-k,j}.$$

The result follows since if $x^iy^j\in\Pi_n$ is a monomial of degree less than $(k,n-k)$ then

$$\begin{aligned} x^iy^j\in\,&\mathrm{span}\{U_i(x)U_j(y),(i,j)<_{\mathrm{td}}(k,n-k),\ i<k\}\\ &\oplus\mathrm{span}\{q_i(x,y)U_j(y),(i,j)<_{\mathrm{td}}(k,n-k),\ i\geq k\}.\end{aligned}$$

□

In a similar way, if

$$\tilde{h}(x,w)=\sum_{j=0}^{M}\tilde{h}_j(x)w^j,$$

is a stable polynomial in w for any $-1\leq x\leq 1$ where $\tilde{h}_j(x)$ are polynomials with real coefficients such that the $\deg\tilde{h}_j(x)\leq\frac{M}{2}-|\frac{M}{2}-j|$ and $\tilde{h}_0(x)=1$, we have:

Lemma 3.4. *For any l such that $\lceil\frac{M-2}{2}\rceil\leq l$ the polynomial*

$$\tilde{q}_l(x,y)=\sum_{j=0}^{M}\tilde{h}_j(x)U_{k-j}(y),$$

is orthogonal to any polynomial in y of degree less than l with respect to

$$d\mu_x(y)=\frac{\sqrt{1-y^2}}{|\tilde{h}(x,w)|^2}dy.$$

Moreover the norm of $\tilde{q}_l$ is independent of x.

Analogous results hold if we order the monomials in the lexicographical or reverse lexicographical orderings. Let $\Pi_{n,m}$ be given as above and order the monomials according to the lexicographical ordering. Let $\kappa=\max\mathrm{degree}(h_i(y))$ and q_r be given by equation (3.2). Then we find

Lemma 3.5. *For* $\lceil \frac{N-1}{2} \rceil \le r \le n$ *and* $0 \le k \le m-\kappa$,

$$p^k_{r,m} = q_r(x,y)U_k(y)$$

is of norm one, of lexicographical degree (r,k)*, and is orthogonal with respect to the measure* (3.5) *to all the monomials in* $\Pi_{n,m}$ *of lexicographical degree less than* (r,k)*. When* N *is even and* $r = N/2-1$ $p^k_{r,m}$ *is given by the above formula up to a multiple.*

Proof. From the definition of $q_r(x,y)$ and the constraints on the degree of $h_i(y)$ we see that $q_r(x,y)U_k(y) \in \Pi_{n,m}$ for $k \le m-\kappa$ and is of lexicographical degree $x^r y^k$. The orthogonality of $q_r(x,y)U_k(y)$ now follows from Lemma 3.1 and the orthogonality of $U_k(y)\sqrt{1-y^2}$ to y^j, $0 \le j < k$. □

The above Lemma shows the remarkable fact that for the weights we are considering if we increase m, $p^k_{r,m}$ does not need to be recomputed.

Likewise with $\gamma = \max \operatorname{degree}(\tilde{h}_j(x))$ we have

Lemma 3.6. *Let*

$$\tilde{p}^l_{n,t} = \tilde{q}_t(x,y)U_l(x).$$

Then for $\lceil \frac{M-1}{2} \rceil \le t \le m$ *and* $0 \le l \le n-\gamma$, $\tilde{p}^l_{n,t}$ *is of norm one, of reverse lexicographical degree* (l,t) *and is orthogonal with respect to the measure* (3.5) *to all the monomials in* $\Pi_{n,m}$ *of reverse lexicographical degree less than* (l,t)*. When* M *is even and* $t = M/2-1$ $\tilde{p}^l_{n,t}$ *is given by the above formula up to a multiple.*

At this point without any other assumptions on σ in both orderings there are an infinite number of missing polynomials. More precisely in the total degree ordering we are unable to compute p^k_n for $k < \lceil \frac{N-2}{2} \rceil$ while in the lexicographical ordering $p^k_{r,m}$ is not given for $r < \lceil \frac{N-2}{2} \rceil$ or $k > m-\kappa$. We now consider a class of weights where we can compute $p^k_{r,m}$ for $k > m-\kappa$.

Let

$$\sigma(x,y) = \frac{4}{\pi^2}\frac{\sqrt{1-x^2}\sqrt{1-y^2}}{|h(z,y)|^2} = \frac{4}{\pi^2}\frac{\sqrt{1-x^2}\sqrt{1-y^2}}{|\tilde{h}(x,w)|^2}\,, \tag{3.6}$$

where $h(z,y) = \omega(z,w)\omega(z,1/w)$, with

$$\omega(z,w) = \prod_{i=1}^{N}(1+a_i zw), \tag{3.7}$$

$x = \frac{1}{2}(z+1/z)$ and $y = \frac{1}{2}(w+1/w)$. Here we assume $0 < |a_i| < 1$, $a_i \in \mathbb{R}$. Thus $\omega(z,w)$ is a stable polynomial, i.e. $\omega(z,w) \ne 0$, $|z| \le 1$, $|w| \le 1$. Since for $|z| = 1 = |w|$ we have $|h(z,y)|^2 = \prod_{i=1}^{N}(1+a_i zw)(1+a_i z/w)(1+a_i w/z)(1+a_i/(zw))$ we see that

$$\tilde{h}(x,w) = \omega(z,w)\omega(1/z,w). \tag{3.8}$$

We begin with

Lemma 3.7. *The polynomial* $w^N\omega(z,1/w)$ *is homogeneous in* z *and* w *of degree* N*. With* $h(z,y) = \sum_{i=0}^{2N} h_i(y)z^i$ *then* $\deg h_i \le N-|N-i|$ *and* $h(z,y)$ *is stable for* $-1 \le y \le 1$, $|z| \le 1$*. The same properties hold for* $\tilde{h}$ *with the roles of* z *and* w *and* x *and* y *interchanged respectively.*

Proof. The homogeneity of $w^N\omega(z,1/w)$ follows from the definition of $\omega(z,w)$. Since $(1+a_izw)$ is stable $g_i(z)=(1+a_izw)(1+a_iz/w)=(1+2a_iyz+a_i^2z^2)$ is stable for $-1\le y\le 1$ and $|z|\le 1$. Furthermore with $g_i(z)=\sum_{j=0}^2 g_{i,j}(y)z^j$ we see that deg $g_{i,j}=1-|1-j|$. The degree properties of $h_i(y)$ now follow since $h(z,y)=\prod_{i=1}^N g_i(z,y)$. An analogous argument holds for $\tilde h$. □

Lemmas 3.5 and 3.6 allow us to compute $p^k_{r,m}$ and $\tilde p^l_{n,t}$ for r and t sufficiently large and $k\le m-N$ and $l\le n-N$ respectively. We now embark on the construction of $p^k_{r,m}$ and $\tilde p^l_{n,t}$ for $m-N<k\le m$ and $n-N<l\le n$ respectively. In order to accomplish this, let $K=\text{span}\{U_n(x)U_m(y), n\ge 0, m\ge 0\}$ i.e. the space of all polynomials in x and y, and $\hat K=\text{span}\{z^{-n}w^{-m}, n\ge 0, m\ge 0\}$. We define a linear map $T:K\to\hat K$ by $T(U_l(x)U_j(y))=z^{-l}w^{-j}$ where $y=1/2(w+1/w)$ and $x=1/2(z+1/z)$. It is not difficult to check that T is a linear one to one mapping from K onto $\hat K$. We begin with the simple

Lemma 3.8. *For fixed n and m let $p(x)$ be a polynomial of degree at most n in x and $q(y)$ be a polynomial of degree at most m in y. Then* $T(p(x)q(y)U_n(x)U_m(y))=p\left(\frac{z+1/z}{2}\right)q\left(\frac{w+1/w}{2}\right)z^{-n}w^{-m}$.

Proof. Suppose deg $p(x)=p$ and deg $q(y)=q$. Then from the recurrence formula for Chebyshev polynomials i.e., $U_{k+1}(t)+U_{k-1}(t)=2tU_k(t)$ we see that $p(x)U_n(x)=\sum_{j=-p}^p a_jU_{n-j}$. From the restrictions on the degree of p it follows that the indexes of the Chebyshev polynomials in the above sum are positive so T can be applied to obtain $T(p(x)U_n(x))=\sum_{j=-p}^p a_jz^{-n+j}=z^{-n}p\left(\frac{z+1/z}{2}\right)$. Since the same is true for $q(y)U_m(y)$ the result follows from the linearity of T. □

The purpose of the map T is that it conveniently keeps track of the highest powers of x and y in terms of z and w.

We now prove.

Theorem 3.9. *Let $p^k_{r,m}(x,y)$, $m-N<k\le m$, with $m,r\ge 2N$ be of norm one, of lexicographical degree (r,k) and orthogonal to all monomials in $\Pi_{n,m}$ of lexicographical degree less than (r,k). Then there exist constants $a_{m-k,j}$, $b_{m-k,j}$, $j=0,\dots,k-(m-N)-1$ such that*

$$\begin{aligned}&p^k_{r,m}(x,y)\\&=\sum_{j=0}^{k-(m-N)-1}\{a_{m-k,j}q_{r+j}(x,y)U_{k-j}(y)+b_{m-k,j}\tilde q_{m+1+j}(x,y)U_{r+k-m-1-j}(x)\}\end{aligned}$$

with $a_{m-k,0}\neq 0$.

Proof. From Lemma 3.1 we see that $\tilde q_{m+1+j}(x,y)$ is orthogonal to all monomials y^i, $0\le i\le m$ and $q_{r+j}(x,y)$, $j=1,\dots,k-(m-N)-1$ is orthogonal to all x^l, $l=0,\dots,r$. Furthermore $q_r(x,y)U_k(y)$ is orthogonal to x^iy^j for $0\le i<r$, $0\le j\le m$ and for $i=r$, $0\le j<k$. Thus the right hand side of the above equation satisfies the required orthogonality conditions. We now show that it is a linear combination of the monomials of degree less than or equal to (r,k) in $\Pi_{n,m}$ ordered lexicographically. Set $S_{r,m-N+1}=q_r(x,y)U_{m-N+1}(y)$, $\tilde S_{r-N,m+1}=\tilde q_{m+1}(x,y)U_{r-N}(x)$, $\hat S_{r,m-N+1}=T(q_r(x,y)U_{m-N+1}(y))$ and $\hat{\tilde S}_{r-N,m+1}=T(\tilde q_{m+1}(x,y)U_{r-N}(x))$. Then

from Lemmas 3.7 and 3.8 we find that

$$\hat{S}_{r,m-N+1} = z^{-r}w^{-(m-N+1)}\sum_{i=0}^{2N} h_i\left(\frac{w+1/w}{2}\right)z^i$$
$$= z^{-r}w^{-(m+1)}\omega(z,w)w^N\omega(z,1/w)$$

which contains $w^{-(m+1)}$. Therefore we must eliminate the w^0 term in $\omega(z,w)w^N\omega(z,1/w)$. From Lemma 3.7 we have that this is associated with z^N. Now

$$\hat{\tilde{S}}_{r-N,m+1} = w^{-m-1}\sum_{i=0}^{2N} \tilde{h}_i\left(\frac{z+1/z}{2}\right)w^i z^{-r+N}$$
$$= w^{-m-1}z^{-r}\omega(z,w)z^N\omega(1/z,w),$$

where equation (3.8) has been used to obtain the last equality. From the definition of $\omega(z,w)$ we see that the coefficient of z^Nw^0 in $z^N\omega(1/z,w)$ is 1 and that $z^N\omega(1/z,w)$ and $w^N\omega(z,1/w)$ are homogeneous polynomials of degree N in z and w. Write

$$w^N\omega(z,1/w) = \gamma_N^1 z^N + \gamma_{N-1}^1 z^{N-1}w + \cdots + w^N$$

and

$$z^N\omega(1/z,w) = z^N + \cdots + \gamma_N^1 w^N.$$

Then $\hat{S}_{r,m-N+1} - k_N^1\hat{\tilde{S}}_{r-N,m+1}$ does not contain $w^{-(m+1)}$ if $k_N^1 = \gamma_N^1$. Thus the polynomial $\hat{p}_{r,m}^{m-N+1} = S_{r,m-N+1} - k_N^1\tilde{S}_{r-N,m+1}$ is of lexicographical degree $(r, m-N+1)$ and is orthogonal to all polynomials of degree less than $(r, m-N+1)$. Note that

$$\hat{S}_{r,m-N+1} - k_N^1\hat{\tilde{S}}_{r-N,m+1} = z^{-r}w^{-m}\omega(z,w)\omega^1(z,w)$$

where $\omega^1(z,w)$ is a polynomial homogeneous of degree $N-1$ in z and w.

To continue, set

$$S_{r,m}^1 = S_{r,m-N+1} - k_N^1\tilde{S}_{r-N,m+1}$$

and

$$\tilde{S}_{r,m}^1 = \tilde{S}_{r-N+1,m} - k_N^1 S_{r+1,m-N}.$$

From the above discussion we see that $T(S_{r,m}^1) = z^{-r}w^{-m}\omega(z,w)\omega^1(z,w)$ where $\omega^1(z,w)$ is a polynomial homogeneous in z and w of degree $N-1$, i.e.,

$$\omega^1(z,w) = \sum_{i=0}^{N-1}\gamma_i^2 z^i w^{N-1-i}$$

with $\gamma_0^2 \neq 0$ since $\hat{p}_{r,m}^{m-N+1}$ is of degree $(r, m-N+1)$. Therefore $T(S_{r,m+1}^1) = z^{-r}w^{-m-1}\omega(z,w)\omega^1(z,w)$. Likewise because of the relationship between $S_{r,m}$ and $\tilde{S}_{r,m}$ we find $T(\tilde{S}_{r,m+1}^1) = z^{-r}w^{-m-1}\omega(z,w)\tilde{\omega}^1(z,w)$ where $\tilde{\omega}^1(z,w) = z^{N-1}w^{N-1}\omega^1(1/z,1/w)$. If we choose $k_{N-1}^2 = \frac{-\gamma_{N-1}^2}{\gamma_0^2}$ then $T(S_{r,m+1}^1 - k_{N-1}^2\tilde{S}_{r,m+1}^1)$ does not contain $w^{-(m+1)}$. Hence

$$\hat{p}_{r,m}^{m-N+2} = S_{r,m+1}^1 - k_{N-1}^2\tilde{S}_{r,m+1}^1$$

is of lexicographical degree $(r, m-N+2)$ and is orthogonal to all monomials in $\Pi_{n,m}$ of lexicographical degree less than $(r, m-N+2)$.

Set

$$S_{r,m}^2 = S_{r,m+1}^1 - k_{N-1}^2\tilde{S}_{r,m+1}^1$$

and

$$\tilde{S}_{r,m}^2 = \tilde{S}_{r+1,m}^1 - k_{N-1}^2 S_{r+1,m}^1.$$

The above discussion shows that

$$T(S^2_{r,m}) = z^{-r}w^{-m}\omega(z,w)\omega^2(z,w)$$

with $\omega^2(z,w)$ a polynomial homogeneous in z and w of degree $N-2$ with the coefficient of w^{N-2} being nonzero. Likewise $T(\tilde{S}^2_{r,m}) = z^{-r}w^{-m}\omega(z,w)\tilde{\omega}^2(z,w)$ with $\tilde{\omega}^2(z,w) = z^{N-2}w^{N-2}\omega^2(1/z,1/w)$. The construction of an orthogonal set of polynomials of the correct degree now follows by induction, that is,

$$\hat{p}^k_{r,m}(x,y) = \sum_{j=0}^{k-(m-N)-1} \left\{\hat{a}_{m-k,j}q_{r+j}(x,y)U_{k-j}(y) + \hat{b}_{m-k,j}\tilde{q}_{m+1+j}(x,y)U_{r+k-m-1-j}(x)\right\}.$$

To complete the proof we show that the norm of $\hat{p}^k_{r,m}$ depends only on $m-k$. From the definition of q and $\tilde{q}$ and Lemma 3.1 we see that

$$\hat{p}^k_{r,m}(x,y) = c_{m-k}U_r(x)U_k(y) + \text{terms of lower lex degree}.$$

From the orthogonality properties of q and $\tilde{q}$ we find,

$$\|\hat{p}^k_{r,m}\|^2 = c_{m-k}\langle \hat{p}^k_{r,m}, U_r(x)U_k(y)\rangle = c_{m-k}\hat{a}_{m-k,0}\langle q_r(x,y)U_k(y), U_r(x)U_k(y)\rangle.$$

By orthogonality $U_r(x)$ may be replaced by $q_r(x,y)$ showing that $\|\hat{p}^k_{r,m}\|^2 = c_{m-k}\hat{a}_{m-k,0}$ which proves the result. $\square$

A similar discussion shows

Theorem 3.10. *Let $\tilde{p}^l_{n,t}(x,y)$, $n-N < t \le n$, $n,t \ge 2N$ be of reverse lexicographical degree (l,t) orthogonal to all monomials in $\Pi_{n,m}$ of reverse lexicographical degree less than (l,t). Then there exists constants $\tilde{a}_{t-l,j}, \tilde{b}_{t-l,j}$, $j = 0,\dots,l-(n-N)-1$ such that*

$$\tilde{p}^l_{n,t}(x,y) = \sum_{j=0}^{l-(n-N)-1} \left\{\tilde{a}_{t-l,j}\tilde{q}_{t+j}(x,y)U_{l-j}(x) + \tilde{b}_{t-l,j}q_{n+1+j}(x,y)U_{t+l-n-1-j}(x)\right\}.$$

The above discussion gives,

Theorem 3.11. *Let $\sigma(x,y)$ be given as in equations (3.6) and (3.7). Let $\Pi_{n,m} = \mathrm{span}\{x^iy^j, 0 \le i \le n;\ 0 \le j \le m\}$ with $n > 2N$ and $m > 2N$. Then for $2N \le r \le n$, $0 \le k \le m$*

$$p^k_{r,m}(x,y) = \begin{cases} q_r(x,y)U_k(y) & \text{if} \quad k \le m-N \\ \sum_{j=0}^{k-(m-N)-1} \{a_{m-k,j}q_{r+j}(x,y)U_{k-j}(y) + b_{m-k,j}\tilde{q}_{m+1+j}(x,y)U_{r+k-m-1-j}(x)\} \\ \qquad \text{if} \quad k > m-N. \end{cases}$$

Likewise for $2N \le t \le m$

$$\tilde{p}^l_{n,t}(x,y) = \begin{cases} \tilde{q}_t(x,y)U_l(x) & \text{if} \quad l \le n-N \\ \sum_{j=0}^{l-(n-N)-1} \left\{\tilde{a}_{t-l,j}\tilde{q}_{t+j}(x,y)U_{l-j}(x) + \tilde{b}_{t-l,j}q_{n+1+j}(x,y)U_{t+l-n-1-j}(y)\right\} \\ \qquad \text{if} \quad l > n-N. \end{cases}$$

4. Recurrence Coefficients

We now examine the consequences for the recurrence coefficients if we assume that the weights are of the form discussed above. From the one variable theory one might expect that the recurrence coefficients have some simple structure. We begin with the total degree ordering.

Theorem 4.1. *Consider the weight given by equation* (3.5)*. For* $\lceil \frac{N-1}{2} \rceil \le n$*,*

$$A_{y,n} = \begin{bmatrix} C_{y,n} & 0 & \cdots & 0 & 0 \\ 0 & 1/2 & & & \\ \vdots & & \ddots & & \\ 0 & & & 1/2 & 0 \end{bmatrix}, \tag{4.1}$$

where $C_{y,n}$ *is a* $\lceil \frac{N-1}{2} \rceil \times \lceil \frac{N-1}{2} \rceil$ *lower triangular matrix with positive diagonal elements*

$$B_{y,n} = \begin{bmatrix} D_{y,n} & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}, \tag{4.2}$$

where $D_{y,n}$ *is a symmetric* $\lceil \frac{N-2}{2} \rceil \times \lceil \frac{N-2}{2} \rceil$ *matrix,*

$$A_{x,n} = \begin{bmatrix} C_{x,n} & 0 \\ 0 & 1/2 I_{n-\lceil \frac{N-1}{2} \rceil, n-\lceil \frac{N+1}{2} \rceil} \end{bmatrix}, \tag{4.3}$$

where $C_{x,n}$ *is an* $\lceil \frac{N+1}{2} \rceil \times \lceil \frac{N+1}{2} \rceil$ *lower Hessenberg matrix with positive entries in the upper diagonal and*

$$B_{x,n} = \begin{bmatrix} D_{x,n} & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.4}$$

where $D_{x,n}$ *is a symmetric* $\lceil \frac{N}{2} \rceil \times \lceil \frac{N}{2} \rceil$ *matrix . Here* $I_{n-\lceil \frac{N-1}{2} \rceil, n-\lceil \frac{N+1}{2} \rceil}$ *is an* $(n - \lceil \frac{N-1}{2} \rceil) \times (n - \lceil \frac{N+1}{2} \rceil)$ *matrix with ones on the* $(i, i+1)$ *entries and zeros everywhere else.*

Proof. If we examine the matrix elements in $A_{y,n}$ we see that for $\lceil \frac{N-1}{2} \rceil \le i$,

$$[A_{y,n}]_{i,j} = \int_{-1}^{1} \int_{-1}^{1} y q_i(x,y) U_{n-i}(y) p^j_{n+1}(x,y) d\mu(x,y).$$

The recurrence formula for the Chebyshev polynomials shows that $y q_i(x,y) U_{n-i}(y) = \frac{1}{2}(p^i_{n-1}(x,y) + p^i_{n+1}(x,y))$ which implies equation (4.1). Equation (4.2) follows in a similar manner. To show (4.3) we write for $\lceil \frac{N+1}{2} \rceil \le i$

$$[A_{x,n}]_{i,j} = \int_{-1}^{1} \int_{-1}^{1} x q_i(x,y) U_{n-i}(y) p^j_{n+1}(x,y) d\mu(x,y).$$

From the definition of q_i we see that $x q_i = \frac{1}{2}(q_{i-1} + q_{i+1})$. So that $x q_i(x,y) U_{n-i}(y) = \frac{1}{2}(p^{i-1}_{n-1}(x,y) + p^{i+1}_{n+1}(x,y))$ which gives (4.3) . Equation (4.4) follows in a similar manner. □

For the polynomials ordered lexicographically we have,

Theorem 4.2. *Consider the weight given by equation* (3.5)*. In the lexicographical ordering we have for* $\lceil \frac{N+1}{2} \rceil \leq n$

$$A_{n,m} = \begin{bmatrix} 1/2 I_{m-\kappa+1} & 0 \\ 0 & C_{n,m} \end{bmatrix}, \tag{4.5}$$

where $C_{n,m}$ *is a* $\kappa \times \kappa$ *lower triangular matrix with positive diagonal entries and*

$$B_{n,m} = \begin{bmatrix} 0 & 0 \\ 0 & D_{n,m} \end{bmatrix}, \tag{4.6}$$

with $D_{n,m}$ *a symmetric* $\kappa \times \kappa$ *matrix. For weights of the form* (3.6)-(3.7) *and* $2N < n, m$ *we have*

$$A_{n,m} = 1/2 I_{m+1} \text{ and } B_{n,m} = 0 \tag{4.7}$$

and

$$\tilde{A}_{n,m} = 1/2 I_{n+1} \text{ and } \tilde{B}_{n,m} = 0. \tag{4.8}$$

Proof. The matrix elements in $A_{n,m}$ are given by

$$[A_{n,m}]_{i,j} = \int_{-1}^{1}\int_{-1}^{1} x p^i_{n,m}(x,y) p^j_{n-1,m}(x,y) d\mu(x,y).$$

Thus equation (4.5) follows from Lemma 3.5. If the weight is of the form (3.6)-(3.7) and $2N < n, m$ then the first part of equation (4.7) follows from Theorem 3.9. The remaining claims follow in a similar manner.

□

5. Examples

In this section we consider several examples including the ones discussed in [5].

5.1. Example 1.

Here we consider the polynomials orthogonal with respect to the probability measure on $[-1,1]^2$

$$d\mu = \frac{4}{\pi^2} w(x,y)\sqrt{1-x^2}\sqrt{1-y^2}\,dx\,dy, \tag{5.1}$$

where

$$w(x,y) = \frac{(1-a^2)}{4a^2(x^2+y^2) - 4a(1+a^2)xy + (1-a^2)^2}, \tag{5.2}$$

and a is real and $|a| < 1$. This measure is of the form equations (3.6) and (3.7) with $N = 1$ and $h(z,y) = 1 - 2ayz + a^2z^2$.

From Lemma 3.1 we know that the polynomials $q_0(x,y) = 1$ and

$$q_k(x,y) = \frac{1}{\sqrt{1-a^2}}\left(U_k(x) - 2ayU_{k-1}(x) + a^2U_{k-2}(x)\right)$$

for $k \geq 1$ are orthonormal with respect to the measure

$$d\mu_y(x) = \frac{2}{\pi}\frac{(1-a^2)\sqrt{1-x^2}}{4a^2(x^2+y^2) - 4a(1+a^2)xy + (1-a^2)^2}dx.$$

Applying Theorem 3.3 shows that for $k \geq 0$

$$p^k_n(x,y) = q_k(x,y)U_{n-k}(y). \tag{5.3}$$

This gives the recurrence coefficients [5]

$$A_{x,n} = \frac{1}{2}\begin{bmatrix} a & \sqrt{1-a^2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$A_{y,n} = \frac{1}{2}\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{bmatrix},$$

and $B_{x,n}$, $B_{y,n}$ are $(n+1)\times(n+1)$ zero matrices.

For the polynomials obtained via the lexicographical ordering Theorem 3.11 shows that for $n, m > 2$,

$$p^k_{n,m} = \begin{cases} q_k(x,y)U_k(y) & \text{for } k < m \\ a_1 q_{n+1}(x,y)U_m(y) + b_1\tilde{q}_{m+1}(x,y)U_{n-1}(x) & \text{for } k = m. \end{cases} \tag{5.4}$$

5.2. **Example 2.** Consider the weight $w(x,y)$ associated with

$$h(z,y) = (1-2bz)(1-2ayz+a^2z^2), \qquad |b| < 1/2, \quad |a| < 1.$$

Lemma 3.1 shows that for $k \geq 1$ the polynomials

$$q_k(x,y) = U_k(x) - 2(ay+b)U_{k-1}(x) + a(4by+a)U_{k-2}(x) - 2a^2bU_{k-3}(x) \tag{5.5}$$

are orthogonal with respect to the probability measure

$$d\mu_y(x) = \frac{2}{\pi}\frac{w(x,y)\sqrt{1-x^2}}{(1-4bx+4b^2)}$$

on $[-1,1]$, where $w(x,y)$ is defined in (5.2). It is easy to see that

$$\int_{-1}^{1} d\mu_y(x) = \frac{1}{1-4aby+4a^2b^2}, \quad |b| < 1/2, \tag{5.6}$$

which is another Bernstein-Szegő weight whose orthogonal polynomials are easily seen to be

$$V_n(y) = \begin{cases} U_n(y) - 2abU_{n-1}(y), & \text{if } |2ab| > 1 \\ 2abU_n(y) - U_{n-1}(y), & \text{if } |2ab| < 1. \end{cases}$$

Hence, applying Theorem 3.3 and defining $q_0(x,y) = 1$ shows that

$$p^k_n(x,y) = \begin{cases} V_n(y), & k = 0, \\ q_k(x,y)U_{n-k}(y), & 1 \leq k \leq n \end{cases} \tag{5.7}$$

give a complete set of orthogonal polynomials in the total degree ordering. It follows that the recurrence coefficients in the total degree ordering are as suggested in [5], i.e. $A_{x,n}$ and $A_{y,n}$ are the same as in the previous example. For $B_{x,n}$ we have

$$B_{x,0} = [b], \quad B_{x,1} = b\begin{bmatrix} 1-a^2 & -a\sqrt{1-a^2} \\ -a\sqrt{1-a^2} & a^2 \end{bmatrix}$$

and for $n \geq 2$, $B_{x,n}$ is the block matrix

$$B_{x,n} = \begin{bmatrix} B_{x,1} & 0 \\ 0 & 0 \end{bmatrix}.$$

The matrices $B_{y,n}$ are identically equal to zero for $n \geq 1$ and $B_{y,0} = [ba]$.

5.3. **Example 3.** For completeness we include also the formulas for the second example considered in [5]. The measure in this case is given by

$$\begin{aligned} d\mu(x,y) =& \frac{2z_0}{\pi^2(x_0-x)} w(x,y)\sqrt{1-x^2}\sqrt{1-y^2}dxdy \\ &+ \frac{2(1-z_0^2)}{\pi} w(x,y)\delta(x-x_0)\sqrt{1-x^2}\sqrt{1-y^2}dxdy, \end{aligned} \tag{5.8}$$

where again $w(x,y)$ is defined in (5.2). Here $z_0 = \frac{1}{2b}$ is real with magnitude less than one, $x_0 = \frac{1}{2}(z_0 + z_0^{-1})$, the first part of the measure is on $[-1,1]^2$, the second is on $\mathbb{R} \times [-1,1]$ and $w(x,y)$ is given by (5.2).

If $|b| < 1/2$ then $\frac{z_0}{\pi(x_0-x)} w(x,y)\sqrt{1-x^2}$ is exactly the Bernstein-Szegő weight discussed in Example 2. When $|b| > 1/2$, h is no longer stable and a modification of the calculation in Lemma 3.1 shows that the polynomials given in (5.5) are orthonormal with respect to the measure

$$d\mu_y(x) = \frac{z_0}{\pi(x_0-x)} w(x,y)\sqrt{1-x^2}dx + (1-z_0^2)w(x,y)\delta(x-x_0)\sqrt{1-x^2}dx.$$

A residue calculation gives,

$$\int d\mu_y(x) = \frac{1}{1+4a^2b^2-4aby}, \tag{5.9}$$

which is the same as (5.6). Hence, it follows that the polynomials given in (5.7) are also orthogonal with respect to $d\mu(x,y)$ in (5.8).

The polynomials and the recurrence coefficients in the lexicographical ordering are given in [5] using a Darboux transformation connecting the two examples.

5.4. **Example 4.** In this example we consider

$$h(z,y) = (1-2a_1yz + a_1^2z^2)(1-2a_2yz+a_2^2z^2), \qquad |a_1|, |a_2| < 1.$$

with corresponding measure,

$$d\mu(x,y) = \frac{4}{\pi^2} w(a_1;x,y)w(a_2;x,y)\sqrt{1-x^2}\sqrt{1-y^2}dxdy,$$

in which $w(a;x,y)$ is defined as in (5.2). Lemma 3.1 shows that for $k \geq 1$ the polynomials

$$\begin{aligned} q_k(x,y) =& U_k(x) - 2a_1a_2(a_1+a_2)U_{k-1}(x) + (a_1^2+a_2^2+4a_1a_2y^2)U_{k-2}(x) \\ &- 2a_1a_2(a_1+a_2)U_{k-3}(x) + a^2b^2U_{k-4}(x), \end{aligned}$$

where $U_{-m-2}(x) = -U_m(x)$, are orthogonal with respect to the measure $d\mu_y(x)$. On the other hand, a simple computation via residues shows that

$$\int_{-1}^{1} d\mu_y(x) = \frac{1+a_1a_2}{(1-a_1a_2)((1+a_1a_2)^2-4a_1a_2y^2)}.$$

This is again a Bernstein-Szegő weight, whose orthogonal polynomials are given by

$$V_n(y) := U_n(y) - a_1a_2U_{n-1}(y), \qquad n \geq 0.$$

Hence, applying Theorem 3.3 and defining $q_0(x,y) = 1$, we see that the polynomials

$$p_n^k(x,y) = \begin{cases} V_n(y), & k = 0, \\ q_k(x,y)U_{n-k}(y), & 1 \leq k \leq n, \end{cases}$$

are orthogonal with respect to the weight function $d\mu(x,y) = d\mu_y(x)\sqrt{1-y^2}dy$.

Remark 5.1. These examples of Theorem 3.3 show that we can derive a complete basis of orthogonal polynomials in total order if N, the degree of $h(z,y)$, is less than or equal to 4. There are other such weight functions, for example, those corresponding to

$$h(z,y) = (1-b_1z)(1-b_2z)(1-2ayz+a^2z^2), \qquad |b_1|, |b_2| < 1, \quad |a| < 1.$$

In this case the integral $d\mu_y(x)$ yields a Bernstein-Szegő weight with two linear factors in y (see Example 2).

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